

# Pedagogical Content Tools: Integrating Student Reasoning and Mathematics in Instruction

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Teaching in a manner consistent with reform recommendations is a challenging and often overwhelming task. Part of this challenge involves using students' thinking and understanding as a basis for the development of mathematical ideas (cf. NCTM, 2000). The purpose of this article is to address this challenge by developing the notion of *pedagogical content tool*. A pedagogical content tool is a device such as a graph, diagram, equation, or verbal statement that a teacher intentionally uses to connect to student thinking while moving the mathematical agenda forward. We tender two examples of pedagogical content tools: Transformational record and generative alternative. These two pedagogical content tools are put forth as instructional counterparts to the Realistic Mathematics Education (RME) design heuristics of emergent models and guided reinvention, respectively. We illustrate the pedagogical content tools of transformational record and generative alternative by drawing on examples from two classroom teaching experiments in undergraduate differential equations.

**Key words:** Classroom interaction, Instructional intervention, Modeling, Oral communication, Postcalculus mathematics, Representations, Teaching knowledge, Teaching practice

Reports from groups such as the National Research Council (1991; 2001), the National Council of Teachers of Mathematics (2000), and the International Commission on Mathematical Instruction (Holton, 2001) have created an influx of attention to the improvement of mathematics teaching. As university teachers and K–12 teachers alike rethink their mathematics teaching practice, they are simultaneously reconsidering important mathematical ideas and determining how to use student thinking and understanding as launching points for the development of these ideas. But how might a teacher continue to develop mathematical ideas with students when their meaningful constructions appear to be inadequate for the

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bigger picture that the teacher envisions? Reflecting on her experiences with third graders, Ball (1993) posed the question in the following way:

How do I create experiences for my students that connect with what they now know and care about but that also transcend their present? How do I value their interests and also connect them to ideas and traditions growing out of centuries of mathematical exploration and invention? (p. 375)

Similar questions have arisen for us in the context of teaching undergraduate mathematics. Our goal in this article is to begin to address these pervasive and deep-rooted questions by developing the notion of *pedagogical content tool* (PCT). A PCT is a device, such as a graph, diagram, equation, or verbal statement, that a teacher intentionally uses to connect to student thinking while moving the mathematical agenda forward. Thus, a PCT involves not only the purposeful activity of relating to student thinking while further developing the mathematics but also the tool or implement that a teacher uses to achieve these goals. We call these *pedagogical content tools* because such teacher interventions suggest the need for the kind of specialized content knowledge that Shulman (1986) referred to as pedagogical content knowledge.

In general, a tool is something that the informed user explicitly recognizes as useful for achieving specific goals (Bateson, 1972; Meira, 1998; Nemirovsky, Tierney, & Wright, 1998; Polanyi, 1958). In every occupation there is a collection of tools that professionals purposefully use. Teaching is no exception. Tools that are at the disposal of teachers include graphs, diagrams, equations, verbal statements, gestures, thought experiments, and so on.

In this article we elaborate on two different types of PCTs that teachers purposefully use to engage students' current thinking while keeping an eye on the mathematical horizon (Ball, 1993). We refer to these two PCTs as *transformational records* and *generative alternatives*. Transformational records are defined as notations, diagrams, or other graphical representations that are initially used to record student thinking and that are later used by students to solve new problems. Generative alternatives are defined as alternate symbolic expressions or graphical representations that a teacher uses to foster particular social norms for explanation and that generate student justifications for the validity of these alternatives. These two PCTs are specifically developed as the teaching counterparts to the Realistic Mathematics Education (RME) instructional design heuristics of emergent models and guided reinvention, respectively. The relationship between these two RME design heuristics and the PCTs of transformational record and generative alternative is elaborated on in the next section.

Knowing when, what kind, and how to use tools such as graphs, diagrams, equations, or verbal statements to connect to student thinking and move the mathematical agenda forward requires a combination of specific content knowledge, general pedagogical expertise, and knowledge of subject matter for teaching. Such blended expertise is part of what teachers need in order to teach (Ball, 1998; Ball & Cohen 1996; Hill, Rowan, & Ball, 2005; Russell, 1997; Shulman, 1986). Mapping out this

enacted expertise, which includes how teachers use tools to proactively support student learning, is an increasingly important line of inquiry (Ball, Lubienski, & Mewborn, 2001; Lampert, 2001).

In relation to this blended expertise for teaching, we have also found in our work with university teachers in differential equations that certain theoretical ideas can be useful orienting heuristics for both research and practice. For example, the constructs of social and sociomathematical norms<sup>1</sup> (Yackel & Cobb, 1996) explicate the function of explanation and justification on evolving student beliefs (Yackel & Rasmussen, 2002), and they offer teachers a structure for reflecting on the nature of their classroom participation structure (Rasmussen, Yackel, & King, 2003; Yackel, Rasmussen, & King, 2000). We make a distinction, however, between PCTs and constructs such as social and sociomathematical norms, the latter of which we refer to as conceptual tools. The term *conceptual tool* comes from Grossman and colleagues' research in English education. Conceptual tools are defined as principles, frameworks, and ideas about teaching and learning that are useful orienting heuristics (Grossman, Smagorinsky, & Valencia, 1999). Conceptual tools, however, do not address the problem of what to do in the classroom. PCTs, on the other hand, speak directly to the problems of teaching mathematics—of how teachers can proactively support their students' learning in the classroom.

More generally, PCTs could be useful in professional development efforts. For example, policy recommendations from the NRC (2001) identify four different promising approaches to professional development, all of which “integrate the study of mathematics and the study of students' learning so that teachers will forge connections between the two” (p. 385). PCTs offer specificity on two ways in which teachers might actually achieve this vision in their daily practice. PCTs may also prove to be a useful component for future models of inquiry-based teaching. As such, this work is both pragmatically and theoretically driven.

## THEORETICAL BACKGROUND

The PCTs developed in this article are two ways in which the RME design heuristics of emergent models and guided reinvention are recast for teaching rather than for instructional design. In a manner consistent with how Simon (1995) examines the ways constructivism might “contribute to the development of useful theoretical frameworks for mathematics pedagogy” (p. 117), we examine ways in which the instructional design theory of RME can similarly contribute. Just as constructivist teaching is a misnomer, we think that RME teaching is a misnomer.

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<sup>1</sup> Social norms pertaining to explanation and justification refer to routine patterns of discursive interaction. Examples of such norms include explaining one's thinking, listening to and attempting to make sense of others' reasoning, indicating agreement or disagreement with others' reasoning, and responding to requests for clarification or challenges. These norms might equally characterize an English class or a history class. Sociomathematical norms, on the other hand, are those aspects of explanation and justification that fall under the purview of mathematics. Examples of sociomathematical norms include what constitutes an acceptable, elegant, or different explanation (Yackel & Cobb, 1996).

Nevertheless, as with constructivism, the instructional design theory of RME can contribute to the development of useful constructs for mathematics pedagogy.

The first RME heuristic that we modify for contributing to theory on teaching is that of emergent models. As described by Gravemeijer (1999), this heuristic can be thought of in terms of a global transition in which students and the teacher develop a *model-of* their informal activity, which gradually develops into a *model-for* more formal mathematical reasoning. This global transition is a process by which a new mathematical reality emerges, grounded in informal and situation-specific activity. Gravemeijer (1999) exemplifies the model-of/model-for transition using a case study of a first-grade class learning to measure. In this measurement sequence, students use a ruler as a model-of iterating some measurement unit, which then evolves into interpreting positions on the ruler as signifying the results of iterating the measurement unit. As the learning tasks are expanded to include incrementing, decrementing, and comparing lengths, students' counting strategies are represented with arcs on an empty number line<sup>2</sup> (Treffers, 1991). The number line then serves as a model-for formal reasoning with number relations.

The model-of to model-for transition, as pointed out by Gravemeijer (1999), is compatible with Sfard's (1991) notion of reification. Connecting the model-of/model-for transition to reification is a strong requirement that typically accompanies extended periods of time. We did not want reification to be a requirement for our notion of transformational record because we wanted a teaching mechanism at a micro, or day-to-day, level (as opposed to thinking about instructional design at the macro level). Thus, the transformational record construct represents a weakening of the emergent model heuristic by not requiring reification and places this idea at the micro level in terms of day-to-day teaching practice.

The second RME instructional design heuristic that we modify is guided reinvention. As described by Gravemeijer (1999), the guided reinvention heuristic outlines a route by which students can develop the intended mathematics for themselves. The emphasis of guided reinvention is on the character of the learning process, rather than on the inventing as such (Freudenthal, 1973). On a continuum of instructional perspectives from pure discovery to pure telling, we view guided reinvention as situated toward the middle of such a continuum. McClain and Cobb (2001) refer to a similar continuum of a teacher's actions ranging from noninterventionist to assuming total responsibility. The PCT of generative alternative, what we propose as a teaching counterpart construct to the RME heuristic of guided reinvention, speaks to how a teacher can actually navigate this continuum.

A typical question that we have encountered when working with teachers is this: "But what if students don't come up with such and such idea or line of reasoning or such and such result?" This most significant question reflects the need for the teacher to play a proactive role in the delicate balance between student invention and direct telling of discipline-specific methods and ways of reasoning, a balance

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<sup>2</sup> The number line is dubbed "empty" because it typically only shows those numbers that students use in their computations.

that respects the intended character of the reinvention process. The notion of generative alternative suggests a means by which teachers can mediate between a noninterventionist form of practice and assumption of total responsibility for explicating certain mathematical concepts and ways of reasoning. The significance of generative alternative for classroom learning is that it serves the function of furthering students' mathematical reasoning and contributes to the ongoing constitution of desirable social and sociomathematical norms regarding explanation and justification.

Transformational record and generative alternative are constructs that explicitly relate teacher actions to the RME instructional design theory and to a perspective on learning that blurs the traditional separation between "the mind" and the things people say and do (Sfard & Kieran, 2001). If we take symbolizing and participation in argumentation as learning, then we need constructs that organize and illuminate features of teaching that promote symbolizing and argumentation. Moreover, constructs about teaching and instructional design ought to be linked to a theory of learning and, conversely, perspectives on learning influence research on teaching and instructional design (Cobb, 2000; Romberg & Carpenter, 1986). In general, RME is compatible with constructivist theories of learning (Gravemeijer, 1994; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). Our particular theoretical orientation is grounded in a version of social constructivism referred to as the *emergent perspective* (Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996). A primary assumption from this point of view is that mathematical learning can be usefully characterized as both a process of mathematical enculturation and as a process of active individual participation.

On the one hand, we work from the premise that meaning is constituted through interaction (Blumer, 1969). Students develop ways of communicating, reasoning, and providing arguments to defend their ideas as they participate in and contribute to the norms and practices of their learning communities. On the other hand, we draw on constructivism (von Glasersfeld, 1995), which emphasizes that learning is a process involving the constant interaction between the learner and his or her environment. An individual's interpretations and construals do not exist apart from the evolving norms and practices of their classroom community. Conversely, communal processes do not exist apart from the interpretations and construals of the active, cognizing individual (Cobb & Yackel, 1996). PCTs offer learner's opportunities for participating in and advancing their own and their communities' mathematical activity.

The work presented in this article contributes more broadly to an emerging body of research (e.g., Boaler, 2003; Bowers & Nickerson, 2001; Lampert, 2001; Lobato, Clarke, & Ellis, 2005; Lovin, Cavey, & Whitenack, 2003; McClain, 2002; McClain & Cobb, 2001; Staples, 2004) that is specifying important aspects of the proactive role of the teacher in creating and sustaining innovative learning environments where students learn mathematics with understanding and/or make shifts in their views about mathematics learning and teaching. For example, Bowers and Nickerson (2001) identified how different patterns of teacher-student interaction fostered a

group of preservice teachers' reorganization of their mathematical ideas and views of effective teaching. This work is significant because rather than pointing to the shortcomings in communication patterns, it specified productive aspects of the teacher's role in supporting the development of students' conceptual orientation to teaching.

McClain and Cobb (2001) highlighted how a teacher's notating of student responses played an important role in constituting the sociomathematical norm of what counts as a different mathematical solution. A specific connection we make to this work is a focus on how a teacher's proactive role in symbolizing and notating can further students' mathematical reasoning and function to promote and sustain desirable social norms pertaining to explanation and justification. Finally, Lobato, Clarke, and Ellis (2005) offered a reconceptualization of the teacher's role regarding the telling of new information. They reformulated telling as "the set of teaching actions that serve the function of stimulating students' mathematical thoughts via the introduction of new ideas into a classroom conversation" (p. 101). The notion of PCT that we advance in this article fits well within this broader recasting of telling in mathematics education.

## METHOD

Data for this analysis come from two different semester-long classroom teaching experiments (Cobb, 2000) in differential equations. Instruction in both classrooms generally followed an inquiry approach, in which important mathematical ideas and methods emerged from students' problem-solving activities and discussions about their mathematical thinking. As such, the learning environments of these teaching experiments provided an appropriate milieu for examining the types of teacher interventions that built on students' ideas and furthered students' mathematical reasoning. The course materials used in these teaching experiments were inspired by the instructional design theory of RME and largely designed in previous teaching experiments.

We refer to the two teaching experiment classrooms as Classroom A and Classroom B. Classroom A consisted of 12 students at a mid-sized public university who were taught by an experienced mathematician with an active research program in partial differential equations. This was his first experience with the RME-inspired instructional materials, although he had taught differential equations for more than 10 years. Moreover, this was his first experience teaching in an inquiry-oriented manner, as most of his prior differential equations teaching had been conducted in a lecture-style format. His interest in participating in the teaching experiment stemmed from his dissatisfaction with lecturing because he recognized that many of his students did not develop deep, conceptual understandings.

Classroom B was taught by the second author of this article. She has a Ph.D in mathematics education and a strong background in mathematics. This was her second time teaching differential equations with the RME-inspired instructional materials and her class contained 45 students. Classroom B took place in a large,

public, urban university with a nontraditional student population, in the sense that the majority of students have returned to school after spending some time in the workforce.

We collected video recordings using two cameras of each class session, retained copies of students' written work, conducted video-recorded interviews with individual students, and held audio-recorded weekly project meetings that included the classroom teacher and at least one other researcher who attended each class session. The weekly project meetings tended to focus pragmatically on both understanding what had transpired in the previous class sessions and planning for subsequent classes. This, at times, included modifying instructional tasks developed in earlier teaching experiments, watching and analyzing videotapes, and discussing the intention and mathematical goals of the instructional tasks. As such, these project meetings often served the purpose of developing possible paths by which learning might take place. These hypothetical learning trajectories (Simon, 1995) were background against which the teacher and students then blazed their own trail (see Nemirovsky & Monk (2000) for a discussion of trail making versus path following).

As Cobb (2000) noted, one of the strengths of classroom teaching experiments is that they offer opportunities to delineate aspects of effective reform teaching because teachers who participate in teaching experiments often become quite effective in supporting their students' mathematical growth. Indeed, the inspiration for PCTs grew out of previous teaching experiments conducted with numerous colleagues.<sup>3</sup> However, the analysis of data from the teaching experiments conducted in Classrooms A and B was essential to sharpen, refine, and develop criteria for the construct of PCTs. In particular, we reviewed all classroom video recordings and made a log of all episodes that fit our criteria (subsequently outlined) for each type of PCT. We then selected exemplar episodes to analyze in detail and to present in this article. Thus, the examples we present are not meant to be representative of every class session but were chosen to illustrate and clarify the two types of PCTs, transformational record and generative alternative.

We identified episodes of a transformational record by the function that the record served in organizing and further developing student's ways of reasoning about particular mathematical ideas. In particular, an episode was characterized as an example of a transformational record when (1) some form of notation (typically informal or unconventional notation) was either used by a student in whole-class discussion or introduced by the teacher to record or notate student reasoning and (2) this notational record was then used by students in achieving subsequent mathematical goals. As argued convincingly by Meira (1998) and others, production or progression in notational use (or symbol use) is dialectically related to student interpretation and sense-making. As such, these two criteria are consistent with our definition of transformational record and directly address students' learning and their construal of meaning.

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<sup>3</sup> These colleagues include Erna Yackel, Michelle Stephan, Karen Allen, Oh Nam Kwon, Wei Ruan, Michael Keynes, and Karen King.

We also developed two criteria for identifying episodes of generative alternatives. Students in both classes were often asked to make graphical or symbolic predictions, and on numerous occasions the teacher invited students to consider alternatives, either student-generated or invented by the teacher, to one of more of their predictions. We coded such episodes as examples of generative alternatives when the alternatives functioned to (1) contribute to the ongoing constitution of social norms pertaining to explanation and (2) elicit or generate justifications for why someone believed particular graphs or symbolic expressions to be mathematically correct or incorrect. That is, these *alternatives generated* explanations and justifications (hence the term *generative alternative*). As demonstrated by Stephan and Rasmussen (2002), students' mathematical learning can be traced by analyzing the evolution of students' explanations and justifications. In this sense, mathematical learning proceeds by argumentation (Lampert & Cobb, 2003; Sfard & Kieran, 2001).

### TRANSFORMATIONAL RECORD

As the person who knows the discipline, a teacher has the obligation of enculturating students into the discourse and conventional representational forms of the broader community while honoring and building on students' contributions. In keeping with the RME design heuristic of emergent models, the purpose of the PCT we refer to as transformational record addresses the important role of the teacher in this process.

#### *Example from Classroom A*

In the example that follows, we first show how the differential equations teacher from Classroom A initially recorded student reasoning in a way that an expert in the subject would recognize as the beginning of a tangent vector field (also referred to as a slope field). As the lesson progressed, this record became a means for students to reason about the form of the analytic expression for a differential equation. In subsequent class periods, tangent vector fields also functioned as a way for students to reason about the space of solution functions<sup>4</sup> (although analyses of these subsequent episodes are beyond the scope of this article). Therefore, it is of high pragmatic and theoretical interest to examine the proactive role of the teacher in the process by which the tangent vector field was transformed from a record of student reasoning to a means for reasoning about other mathematical ideas.

The example we discuss from Classroom A occurred on the second day of class. Students were asked to make predictions about the shape of a population versus time graph for a single species that reproduces continuously and has unlimited

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<sup>4</sup> Reasoning about the space of solutions includes, for example, justifying the shape of graphs of exact solutions, developing arguments for why graphs of solutions to autonomous differential equations are horizontal shifts of each other, and justifying why approximate solutions would or would not be underestimates or overestimates to an exact solution.



resources (no differential equation was provided). The typical response to this task was an exponential or quadratic-like shaped graph (see Figure 1a) positioned above the  $t$ -axis.

The first topic of conversation initiated by the teacher about students' graphical predictions was whether or not the initial slope at  $P = 10$  and  $t = 0$  should be zero or have a positive value. The first student to speak up on this issue argued that the slope should be positive. The teacher did not at this time notate this student's idea with a tangent vector having positive slope (such as that shown in Figure 1b), but rather led a whole-class discussion about the initial slope issue.

The fact that the teacher withheld making a record of a student's initial response is significant for two reasons. First, it allowed for alternative viewpoints to be expressed, and in the process important mathematical issues were discussed.<sup>5</sup> Second, it enabled other students in the class to take ownership of the positive slope idea. Thus, when the teacher eventually did draw, with positive slope, a tangent vector at  $P = 10$  and time  $t = 0$  (see Figure 1b), the notational record served the function of recording the taken-as-shared reasoning of the classroom community. In this case, the teacher's proactive role in the notating process was far more complex than simply providing a record that fit with a student's thinking for the purpose of introducing a particular conventional mathematical inscription. It also involved an intentional effort to establish a community of learners, one in which students explain their reasoning and listen to and make sense of others' ideas.

Subsequent topics of conversation about students' graphical predictions for population over time included how the rate of change at this initial point compared to the rate of change at a later time, how (and why) the rate of change would compare to the other rates if the initial population was greater than what was originally sketched (e.g., if at  $t = 0$ ,  $P = 20$ ), and what the rate of change would be if the popu-

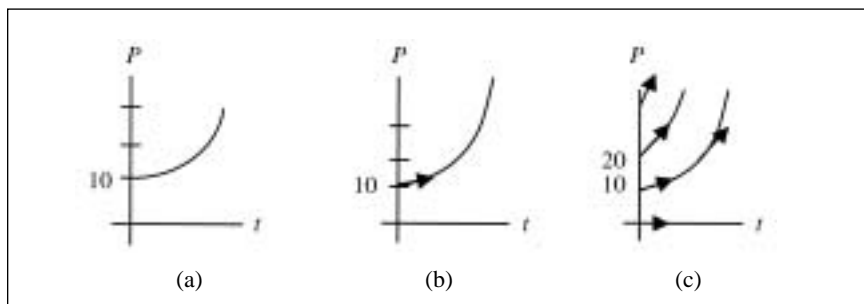


Figure 1. Population versus time graphs and records of student reasoning about rate.

<sup>5</sup> These issues included how the initial slope relates to the assumption of continuous reproduction, whether or not this situation has a history, and what it means in term of motion to have an instantaneous rate of change.

lation at time  $t = 0$  was  $P = 0$ . As was the case with the initial tangent vector record of students' reasoning, these additional conversations provided further opportunities for the teacher to continue to record students' reasoning with additional tangent vectors, such as that shown in Figure 1c. We refer to Figure 1c as an emerging tangent vector field because it is a consequence of classroom discourse and it is beginning to resemble what an expert in differential equations would recognize as tangent vector field.

Class discussions surrounding students' graphical predictions for the population versus time scenario and the resulting records of this reasoning resulted in three important advances. First, these discussions brought forth conceptions of rate of change as a ratio and as an instantaneous quantity. Second, they opened a space for exploring additional mathematical ideas. Third, they provided an opportunity for recording students' reasoning in a way in which the conventional inscription of a tangent vector field began to take form. The point we want to make next is that the tangent vector field, which was initially a record of students' reasoning, subsequently became a means for reasoning about the symbolic form of the rate of change equation. Thus, a transformation in the record took place.

A critical idea that facilitated this transformation emerged out of students' arguments for why, if the initial population is 20 instead of 10 at time  $t = 0$ , the initial slope would be the same as the slope corresponding to  $P = 20$  on the curve that starts at  $P = 10$  at time  $t = 0$  (see Figure 1c). Students' reasons for this horizontal invariance in slope relied on their imagery of the scenario. The basic argument put forth by the class was that it did not matter whether you called time zero Wednesday, Friday, or Labor Day, the population of fish would be increasing at the same rate for a given initial population. What mattered was the number of fish; what you decided to call time zero was arbitrary. As shown in Figure 1c, this line of reasoning was recorded by the teacher with two different tangent vectors that had the same slope at  $P = 20$  but at two different  $t$ -values.

All the mathematical work up to this point occurred without a symbolic expression for the differential equation. Developing the symbolic expression was the next task in the sequence of instructional activities. The task began with the teacher inviting students to consider whether the rate of change,  $dP/dt$ , should depend explicitly on just the population  $P$ , just time  $t$ , or on both  $P$  and  $t$ . That is, if  $dP/dt = \text{something}$ , what should the "something" consist of? Conceptually, this tends to be a challenging task for students for two reasons. First, students need to explicitly distinguish between the rate of change in a quantity and the quantity itself. Second,  $P$  stands for both an unknown function and a variable in the rate of change equation (Rasmussen, 2001). Reasoning about what the explicit variables are in a rate of change equation involves conceptualizing rate as a function, which is cognitively more complex than conceptualizing rate of change as the slope of the tangent line at a point.

This conceptual complexity is reflected in the following student excerpt,<sup>6</sup> in which a student argued that, in order to have an exponential shape for the graph of  $P$  versus

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<sup>6</sup> All student names in this article are pseudonyms.

$t$ , the rate of change equation should depend on both the size of the population and the time:

*John:* Well. It seems like it would be really difficult to get the shape of that graph without using the  $e$  [exponential] function. [*Teacher:* OK, so?] And, from past experience, it seems like when you do that and when you're talking about exponential growth that  $t$  is always a factor. Of, like,  $t$  is always part of the exponent when you're talking about population growth. So, I would have to assume that the size of the population and the time in which it's occurred both are going to be significantly relevant.

John remarked that it would be difficult to get the proposed graphical shape without using the exponential function. One way to interpret this remark is that John is not making explicit the distinction between  $P$  as a function and  $P$  as a variable in the rate of change equation. As the whole-class discussion continued, another student, Bill, refuted John's line of reasoning and pointed to the emerging tangent vector field (which was still on the chalkboard) to support his argument. In other words, the previous record of student reasoning shifted function and served as a means for reasoning why the rate of change equation should depend explicitly on just  $P$ .

*Bill:* OK. We're trying to find what the rate of change is. This differential should tell me the rate of change. That's the question. The something that is the right side of this, uh, the graph, or the right side of the [rate of change] equation. When we looked at our [ $P$  versus  $t$ ] graphs, we all agreed that, when the population reaches a certain size, all the rates of change are going to be the same. Doesn't matter what time they reach that, that change.

Notice that Bill's argument relies on the previous conclusion that "we all agreed that, when the population reaches a certain size, all the rates of change are going to be the same." This statement is significant because horizontal invariance of slopes now becomes the basis from which Bill argues that the rate of change equation should only depend on  $P$ , supporting our claim that the initial record is transformed into a means for reasoning about a different mathematical idea. As Bill continued to discuss his reasoning, the teacher, without comment, added additional tangent vectors to the inscription on the board, as illustrated in Figure 2. These additional markings served to record Bill's thinking and to further develop the emerging tangent vector field into an increasingly conventional looking representation.

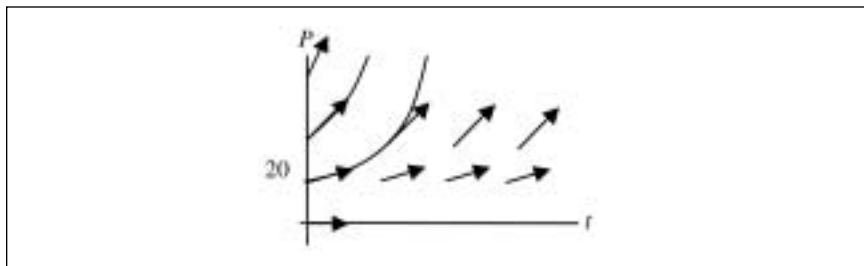


Figure 2. Emerging tangent vector field.

*Bill:* We're just looking to find out what the rate of change is at the population of 40, at the population of 60. So, the only thing that matters was that we know what the population is. It didn't matter that it took an hour for, after, you know, when you start out. Let's call it at 10 population. It didn't matter that the population A took an hour to get there. Only that it is there. Now that rate of change is such and such. It didn't matter that it took population B a half hour to get there. The point is it's at that number. Now the rate of change is such. Same with the next one. So that's why I think that the only thing that's important for you to find the rate of change is to know what the population is. I mean, then you can add some constants to it, whatever you want. But of  $P$  and  $t$ , only  $P$  of those two is important.

As Bill articulated in his argument, the idea and record that for a set population value the rate of change is invariant across time, was transformed into a means for reasoning that the differential equation (that is, the rate of change equation) should depend only on  $P$  and not on time  $t$ . Another point we want to make with this example is that the initial record did not remain an immutable object. As students progressed in their conceptualizations and developed new mathematical insights, the teacher took a proactive role in reshaping the initial record of tangent vectors in a way that fit student reasoning and increasingly moved toward what an expert would recognize as a conventional tangent vector field.

The teacher's proactive role in initiating and furthering the tangent vector field record of student reasoning also included making important decisions about when to *withhold* making a record of student reasoning. As illustrated in this example, the teacher initially chose not to make a tangent vector record of the reasoning expressed by the first student. Instead he acted in accordance with an expectation that students listen to and attempt to make sense of others' reasoning and withheld notation (and he withheld making any evaluative remarks), which, in addition to promoting certain social norms, created an opportunity for the classroom community to develop a line of reasoning for which the emerging tangent vector field was a fit. The fact that this emerging tangent vector field then served as a means for students to reason about what the differential equation should explicitly depend on exemplifies the PCT we refer to as transformational record.

### *Example from Classroom B*

In this example, we show how the teacher introduced informal notation signifying increasing and decreasing solutions to a differential equation as a written record of student reasoning. The teacher's initial record of student reasoning subsequently became a means for students to reason about more complex mathematical ideas on a latter task.

Our example begins on the 5th day of class in which students analyzed a differential equation that reflected a limited growth population assumption. In particular, students were invited to decide if the graph of the oscillating solution shown in Figure 3 was a reasonable graph of a solution to the differential equation

$$\frac{dP}{dt} = 0.3P \left( 1 - \frac{P}{12.5} \right).$$

After students discussed the plausibility of an oscillating solution graph in small groups, the teacher elicited explanations and justifications in support of or against the oscillating graph. The whole-class discussion began with Aimee arguing against the possibility for the oscillating solution graph.

*Aimee:* Um, if we graph that function there [pointing to the differential equation], this cannot be the graph. But if we say that 12.5 is like the number of fish that we can support with our resources and that if we can exceed the number of fish. Like if we can kind of increase the population somehow above 12.5, then it's going to start decreasing. Do you know what I'm saying?

*Teacher:* I think I know what you're saying. Does everyone else know what Aimee is saying?

At this point, the teacher had not recorded any particular part of Aimee's explanation but rather choose to redirect Aimee's question, "Do you know what I'm saying?" back to the whole class, acting in accordance with the social norm that students listen to and make sense of others' ideas. As it happened, Aimee spoke up to further clarify her reasoning.

*Aimee:* For this particular one, this couldn't be, in my opinion this is not the graph. Because at 12.5 it's just going to be a straight line. But if we . . .

*Teacher:* [Moving to the chalkboard] Hmm . . . I'm just going to write down your ideas. Keep going.

*Aimee:* But if we say that 12.5 is the number of fish that we can support with our resources and that we somehow manage to increase our population [above 12.5], that means that our slope is going to become negative. It's going to start decreasing. And then again increasing . . . [Trails off]

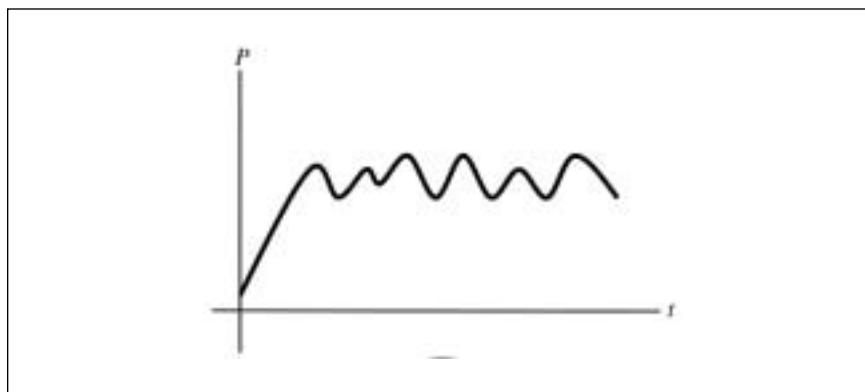


Figure 3. Possible graph of solution to  $\frac{dP}{dt} = 0.3P \left( 1 - \frac{P}{12.5} \right).$

As Aimee spoke, the teacher recorded her reasoning on the oscillating solution graph as seen in Figure 4. The downward pointing arrow was the teacher's record of Aimee's reasoning about the decreasing population, and the upward pointing arrow was the teacher's record of Aimee's reasoning about the increasing population.

*Teacher:* So are you saying something like that above 12.5, based on this rate of change equation, our population will be decreasing?

*Aimee:* Yes.

*Teacher:* OK, so I'm just using these arrows to represent what you said.

*Aimee:* The bottom graph [points to the oscillating graph] could be the more realistic one, like if we were going to include all the other factors, but for that equation there, at 12.5, it's going to [be] more like a straight line.

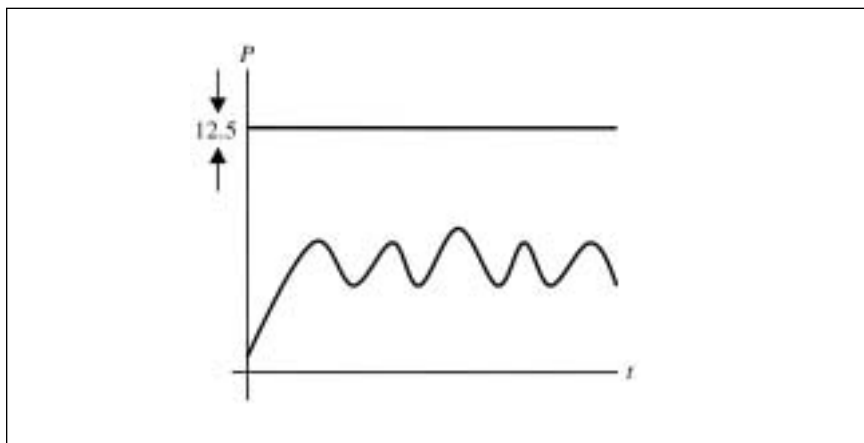


Figure 4. Record of Aimee's reasoning.

We note here that Aimee, not the teacher, made the distinction between reasoning with the rate-of-change equation and reasoning with the imagined real-world setting of a fish population with limited resources.

We claim that the teacher could have pursued a number of different paths at this point in the discussion. Some of these options include asking Aimee to clarify *what* was decreasing and increasing (The number of fish? The rate of change? Something else?) or inviting the class to reflect on the feasibility of the oscillating graph in light of the differential equation (rather than the population situation). We make no value judgments about the teacher's decision not to pursue these topics of conversation, but rather we point to the fact that what the teacher strategically chose to do was to inscribe certain parts of Aimee's argument, specifically, Aimee's statements about increasing and decreasing populations.

Further, we do not claim that what the teacher did was a perfect match with Aimee's reasoning, but rather, paraphrasing von Glasersfeld (1995), it was a viable fit. In addition to being compatible with Aimee's ideas, the teacher's up and down arrow notation is significant because it reflects an eye toward moving students' mathematical reasoning forward. In particular, subsequent mathematical activity involving such an inscription, which one who is well versed in differential equations recognizes as the beginning of a phase line, may offer students a means to conceptualize the two-dimensional solution space in one dimension, encourage students to think more holistically about the solution space, and present opportunities to further contribute to the ongoing constitution of the social norm that students' routinely provide explanations and justifications for their conclusions. Consistent with the language that was used in this class, the teacher's initial record of Aimee's thinking is referred to as a "flow line." In subsequent class sessions, the term flow line was replaced with the conventional terminology of "phase line."

In the next part of the episode, the teacher asked other students for their interpretation in order to gauge whether or not (and why) the horizontal line at 12.5 made sense to them. We liken the teacher's action here to the way in which researchers often ask participants to read and review their interpretations in order to corroborate conclusions, commonly referred to as member checking (Stake, 1995).

*Jake:* I think that we need to consider a couple of things when we look at this. One is that  $P$  is always going to be greater than zero because you can't have a negative population. Since  $P$  is always going to be greater than zero as far as the direction of our rate of change, we can ignore that first term,  $0.3P$  because that's always going to be positive. So we're only looking at our second term,  $1 - (P/12.5)$ . If we just take points along our population, like our initial population's 3,  $3/12.5$  is less than 1, so we're going to have positive growth. We're going to continue to have positive growth until . . . so let's say, a little bit longer, we have 6 fish, we still have positive growth, we have 8 fish, we still have positive growth. Then let's say there's a large . . . all of a sudden we have 14 fish, all of a sudden. Now we're at negative growth. Our growth is going to decrease until it gets below 12.5 again, then we'll, then . . . it's going to fluctuate up and down like that.

It is important to note that the teacher asked Jake to comment on Aimee's *idea*, not the record of Aimee's idea, thus acting in accordance with the social norm that students listen to and make sense of others' ideas and reflecting a theoretical position that meaning is not located in a particular inscription (Meira, 1998). Although Jake did not actually comment on the issue surrounding 12.5, his discussion of positive and negative growth is consistent with the flow line notation (i.e., the up and down arrow record of Aimee's reasoning).

In a subsequent class session, the teacher selected a task in which students were asked to compare two solutions to the differential equation  $dP/dt = 0.2P$ , one with initial condition  $P(0) = 1$  and the other with initial condition  $P(0) = 3$ . Students were asked to physically demonstrate their comparison of the two solutions by moving two cubes along a vertical line with arrow heads, similar to the teacher's initial record of Aimee's reasoning. As shown in Figure 5, a partial slope field was located next

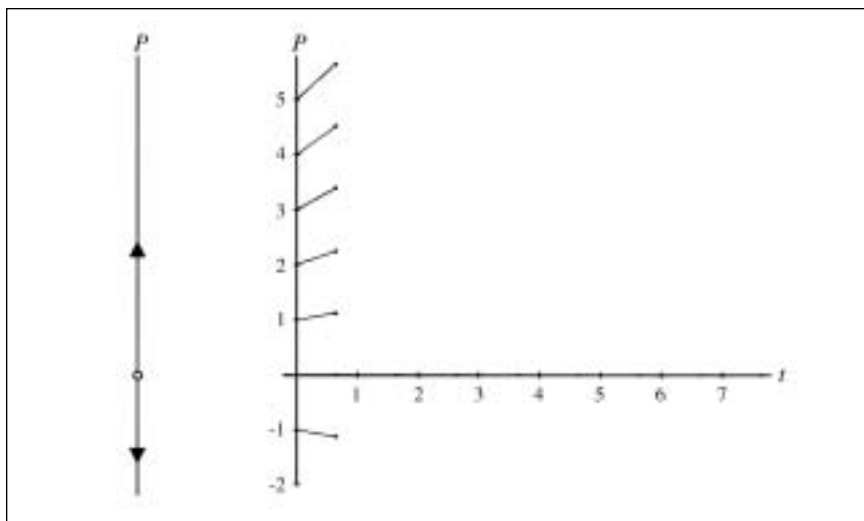


Figure 5. Reasoning about solutions and flow lines.

to the flow line to help support student reasoning that the flow line is a one-dimensional representation of the two-dimensional solution space.

After the students had worked on this task in small groups, the teacher solicited a volunteer to present his or her group's thinking at the overhead projector. Donald volunteered to demonstrate the movement of the two cubes, making reference to the teacher's flow line notation as well as the slopes at the initial time.

*Donald:* At zero there would be no change. You would use the flow line as an interval. Everything above zero is a positive change and everything below zero is a negative change. So what happens to the two cubes at 1 and at 3? Well the first cube [at 1] is going to be moving at a slower rate than the cube that is at 3 because the rate of change at 1 is going to be smaller than the rate of change at 3.

*Teacher:* Can you show the two cubes moving? [Donald then shows the two cubes moving up the flow line, with the distance between the cubes becoming greater the further up the line he moves them.]

Donald's activity with the flow line and his comments are consistent with and extend Aimee's initial reasoning (which, the reader will recall, gave rise to the flow line notation). In particular, Donald stated that "everything above zero is a positive change and everything below zero is a negative change." This reasoning relates to an upward pointing arrow above zero and a downward pointing arrow below zero and is consistent with Aimee's comment that the population was decreasing above 12.5 and increasing below 12.5 on the previous task.

Furthermore, Donald's discussion of the two cubes moving at two different rates is evidence that the flow line, which initially signified direction only, grew into a dynamic object as students pursued their goal of comparing two solutions with



different initial conditions. In other words, Donald's discussion of the two cubes moving at different rates indicates a transformation of the teacher's initial record of student thinking into a means for reasoning about the changing rates of change of the cubes. Another student, Bruce, makes this transformation explicit in his comments that we discuss next.

During the discussion of Donald's ideas, Bruce commented that it would seem to make more sense to draw arrow heads of varying size since the rate of change is greater the higher up the vertical line the block begins. Bruce's modification of the flow line is further evidence that the initial record of Aimee's reasoning was functioning as a means for comparing changing rates of change of solutions.

*Bruce:* Well to me having just the single arrow pointing implies that the rate of change would be the same across all [numbers]  $P$ . It implies that somehow it's just one rate of change up all the way as opposed to having different size arrows as you move up. [He gestures, with his hands, arrow heads getting bigger and bigger as his arms move straight up.]

*Teacher:* So Bruce, you're saying that maybe this single arrow might be just one rate of change. How would you indicate a bigger rate of change?

*Bruce:* By a bigger arrow.

Based on Bruce's gesturing, the teacher drew a series of arrow heads getting bigger and bigger as they were placed higher up on the flow line. There are two significant consequences of Bruce's comments. First, Bruce's statement that "having just the single arrow pointing implies that the rate of change would be the same across all  $P$ " indicates Bruce's belief that the teacher's original inscription is insufficient for the type of reasoning Donald put forth in his discussion. In particular, Bruce suggested that the single arrow on the flow line implies a single rate of change, and he argued that the changing rate of change warrants a modification of the original inscription.

Second, Bruce used the teacher's original record of student thinking as a means for reasoning about the changing rates of change of solutions. The flow line became a dynamic object for him, dynamic in the sense that as you move up the line, the rate at which you move changes. Bruce clearly agreed with Donald's idea that the cube at  $P = 1$  moves slower than the cube at  $P = 3$  and he made this reasoning more explicit by suggesting using arrow heads of varying size.

In this episode, we illustrated how the teacher's informal notation was transformed into a dynamic object as students achieved their goal of comparing solutions with two different initial conditions. That is, the teacher's initial record of Aimee's thinking functioned as a means for subsequent student reasoning. Bruce's suggestion of drawing arrow heads with different sizes implies that the teacher's original notation helped him to reason in increasingly sophisticated ways about rates of change, and in the process the teacher's initial record of Aimee's thinking took on new meaning. The teacher's proactive role in this episode included (1) strategically choosing to notate student reasoning in a way that was at once compatible with the ideas expressed and had the potential to lead to the phase line notation used by the mathematical community and (2) facilitating discourse that supported the norm that students explain their reasoning and try to make sense of others' reasoning.

## GENERATIVE ALTERNATIVES

We now illustrate the PCT of generative alternative, the sister construct to the RME heuristic of guided reinvention. Generative alternatives highlight the proactive role of the teacher in balancing between a noninterventionist form of practice and assumption of total responsibility for explicating certain mathematical concepts and ways of reasoning. As we will argue in the two examples, the significance of generative alternative is that it functions as a teacher-driven mechanism to support students' evolving mathematical reasoning while contributing to the ongoing constitution of productive social norms pertaining to explanation and justification.

*Example from Classroom A*

This example centers on a fairly standard modeling task in which a rate of change equation is developed to predict future amounts of salt in a very large tank. The teacher initiated the task by setting up the following scenario (illustrated in Figure 6):

A very large tank initially contains 15 gallons of saltwater containing 6 pounds of salt. Saltwater containing 1 pound of salt per gallon is pumped into the top of the tank at a rate of 2 gallons per minute, while a well mixed solution leaves the bottom of the tank at a rate of 1 gallon per minute.

We outline classroom interactions in this example in four phases. In phase 1, the teacher invited students to make graphical predictions for the amount of salt in the tank over time. In phase 2, students were asked to decide if the rate of change equation for this situation should explicitly depend just on the amount of salt  $S$  in the tank, on just the time  $t$ , or on both  $S$  and  $t$ . In phase 3 of the task, the class modeled

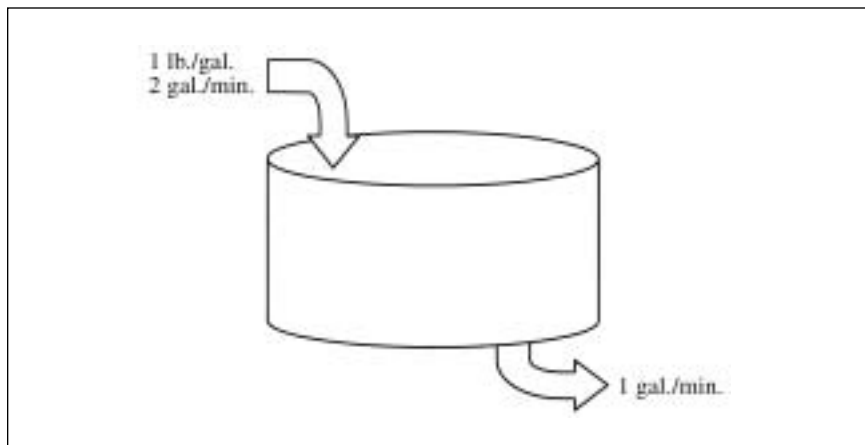


Figure 6. A salty-tank scenario.

this situation with a rate of change equation based on the physical law  $dS/dt = \text{rate of change in} - \text{rate of change out}$ . Finally, phase 4 moved toward finding approximate and exact solutions to the rate of change equation from phase 3 to figure out how much salt is in the tank when the tank is full if the tank has a maximum capacity of 50 gallons.

*Phases 1 and 2: Generating predictions.* The proactive role of the teacher involving generative alternatives occurred in phase 3 of this episode and we therefore focus our analysis of classroom events in that phase. In order for this discussion to make sense to the reader, however, we first summarize what occurred in the first two phases.

The fact that the teacher first invited students to generate predictions of the scenario prior to developing an analytic expression for the differential equation is consistent with his typical pattern of engaging students' sense making by generating qualitative/graphical predictions. These predictions then served as a foundation for subsequent exploration, mathematization, and generalization (Rasmussen & Ruan, in press). The vast majority of students predicted that the amount of salt would continue to increase over time. As explained by one student, "You'd still be losing 1 pound itself per minute while you gain 2 pounds of salt. So, you're always going to be increasing a pound of salt." The point of inviting students to make such predictions (with explanations) was not necessarily to achieve perfectly correct responses but rather to engage students in thinking about the situation in order that they might be more intellectually invested in subsequent parts of the task.

In phase 2 of the task, the teacher continued to engage students in making predictions (with reasons), but this time the predictions were of an analytic, rather than graphical, nature. The majority of students reasoned that the rate of change equation should depend explicitly on both  $t$  and  $S$ . Despite this reasoning,  $S$  tended not to be used as a variable in students' rate of change equations. Instead, when formulating their rate of change equations, students wrote the amount of salt in terms of an expression involving time  $t$ . This is not surprising, however, given the conceptual complexity of reasoning with and about  $S$  as both an unknown function of the amount of salt and as a variable in the differential equation (Rasmussen, 2001; Stephan & Rasmussen, 2002). As evidenced by the discussion regarding the expression for the rate of change out, this conceptual difficulty played itself out by the teacher assuming greater responsibility for developing at least a portion of the differential equation.

*Phase 3: Formulating a rate of change equation.* Consistent with the RME heuristic of guided reinvention, the intention of this phase of the task was to provide students with an opportunity to participate in developing a rate of change equation. The idea behind generative alternative is that by intentionally engaging students in analyzing different possibilities, in this case, different symbolic expressions, the teacher provides occasions for students to generate explanations and justifications, which serve to continually constitute the norm that students offer explanations and justifications while moving mathematical ideas forward.

Recall that the task in this phase was to create for the salty tank situation a rate of change equation,  $dS/dt = \text{rate of change in} - \text{rate of change out}$ . Following the general agreement that the rate of change should include both  $S$  and  $t$ , students worked in small groups, followed by whole-class discussion about their initial ideas for the differential equation. In a spirit of joint inquiry in which ideas were shared and debated, this pattern of small-group work followed by whole-class discussion was repeated twice before the final form of the differential equation

$$\frac{dS}{dt} = 2 - \frac{S}{15 + t}$$

was reached. Cycling between small-group work and whole-class discussion was facilitated by generative alternatives and contrasts with a traditional instructional approach in which the teacher might take total responsibility for developing the differential equation and/or solve a similar problem and then ask students to imitate or repeat the process. We illustrate the role or function of generative alternatives in an annotated transcript centering on the expression for the first part of the rate of change equation, the rate in. The development of the expression for the rate out had a similar character, although the mathematical issues that arose were more complex. Thus, for illustrative purposes, analysis of the development for the rate in expression suffices.

The first of three whole-class discussions focused on student ideas for what the expression for the rate in should be. The conversation began with the teacher asking students to share their ideas for the symbolic form of the rate of change in. Two different ideas for the rate of change in were expressed by students, either  $2$  or  $2t$ . The teacher capitalized on these student-initiated alternatives to continually constitute the social norms that students are to explain and justify their reasoning and to try to make sense of others' reasoning.

*Teacher:* Let's get some ideas, all right? Now. We know they are, they're two parts. The rate of change, the rate that the salt flows in and minus the rate of salt flow out. So we just need to figure out which, for each part. Um. OK. Adam and Kalub? What do you think? How about let's look at the first part? The rate of change that the salt flows in?

*Adam:*  $2t$ .

*Teacher:* [Writes  $dS/dt = 2t$  on the board.] Any different ideas?

*John:* Just  $2$ .

*Teacher:* Just  $2$ ? [Writes  $2$  above  $2t$ .]

*Bill:* I think you have to have  $t$  in that part because you don't know how much over time. Like in 5 minutes, you're going to have 10 gallons flowing in.

*Teacher:* Uh, let me hear, Lonnie? Sean?

*Sean:* I think it would have to be  $2t$ .

*Bill:* [Softly to his group, Jerry and Robert] Should it just be  $2$ ?

Note that after Bill's spontaneous contribution the teacher did not evaluate or interpret his reasoning but rather continued to involve other students in sharing their ideas. As Bill's previous comment and Jerry and Kevin's analysis below suggest, having choices from which to decide tends to open up spaces for providing justi-

fications. It has been our experience that students are more likely to offer justifications when there is a choice to be made. As the discussion continued, the teacher shifted from soliciting different responses to soliciting arguments for why one choice over the other makes sense.

*Teacher:* OK. Now, let's hear your arguments for why  $2t$  or why 2.

*John:* Because, if it was just 2, then you'd be saying that you only added 2 pounds of salt for the whole time that the whole thing went on. But, at  $2t$ , as long as you're doing  $t$  in minutes, dealing with  $t$  minutes, you're saying that every minute it changed by a rate of 2. [*Bill:* Right.] So at 2 minutes you added 4 pounds of salt, at 3 minutes you added 6 pounds of salt.

*Teacher:* So. You're saying that this [points to the expression  $2t$ ] is the amount of salt you're putting into the tank after  $t$  minutes?

*Students:* Uh, huh. Right. Yes.

*Kevin:* Also, think of it this way, that at  $t$  zero, all you have to have is .4, because that's the solution at  $t$  zero.

*Teacher:* Did you [referring to John] change your mind? Because, initially you said it was 2. Did you change it to  $2t$  also?

*John:* Yeah.

*Teacher:* Everybody thinks that it's  $2t$ ?

*Bill:* Uh, huh.

Up until this point, the teacher had been capitalizing on the alternatives of 2 and  $2t$  to proactively support students in generating justifications. In some sense, we might say that the teacher is now in a bind because although his proactive effort at generating justifications has been profitable, many students are now swayed that the expression should be  $2t$ , which is actually incorrect. The teacher next began to assume more responsibility for the mathematical content and direction of the discussion, representing a slide along the noninterventionist-total responsibility continuum. In particular, the teacher recognized that students were not making a conceptual distinction between rate of change in the amount of salt and amount of salt and brought this issue up directly. Identifying and being sensitive to the conceptual issues that students are working on is, as argued by Schifter (2001), a critical ability in being able to build on and extend students' thinking.

*Teacher:* OK. Now. We look for the rate of change for the salt flow in. And, you told me that after  $t$  minutes there are  $2t$  pounds of salt flowing into tank. Is that the rate of change?

*Students:* No. No. It sounds like the salt. Pounds of salt.

*Teacher:* What's the difference?

*Heath:* There [referring to  $2t$ ], at say 3 minutes, you're adding 6 pounds of salt, not 2 pounds. You're always still adding 2 pounds of salt. So time doesn't have anything to do with that. [*Bill:* Right.] Because at time 3 minutes you're saying that you're adding 6 pounds of salt. But in a sense you've added 6 pounds of salt, but you're not adding at that time 6 pounds of salt. Your slope of your slope field is changing with that [referring to  $2t$ ]. It [the rate] should always be 2.

Although his reasoning is not entirely clear,<sup>7</sup> we suspect that Heath's reasoning stemmed from his and his group's prediction of the graph of the amount of salt over time (phase 1). In support of this interpretation, recall that the vast majority of students predicted that the graph would increase over time, perhaps with an eventual constant positive slope, which then might not be consistent with a slope field corresponding to a rate in of  $2t$ . In any case, the teacher did not pick up on this contribution but did pick up on the distinction Heath made between rate of change and the amount of salt.

*Teacher:* OK, so this  $[2t]$  is not the rate of change. This is actually the amount of salt after  $t$  minutes.

*John:* Yeah. Which its derivative is 2.

*Teacher:* So you [John] change back to 2?

*John:* Yeah. [Laughter from several class members.]

*Teacher:* Well tell us a bit more, why you changed back to 2.

*John:* Why? Because  $2t$  is the equation for the amount of salt in the water at any time. And so the derivative of that *is* the rate of change, which is 2.

The conversation continued as the teacher requested that other students respond to the reasons that some students expressed in support of 2 as the rate in and to explain their position as to whether (and why) they changed their mind. As evidence that these actions on the part of the teacher were productive, we note that Robert, who had yet to speak up in class, commented that "Well, he [John] brings up a good point; we actually want the rate of change of salt."

What makes this episode an example of generative alternative is not simply the fact that the teacher focused the whole-class discussion on two alternatives for the rate in, namely, 2 or  $2t$ . We see this as an example of generative alternative because of the way the alternatives functioned for maintaining social norms and for advancing mathematical ideas. The mathematical idea that was advanced through students' explanations and justifications was the explicit distinction between rate of change in a quantity and the quantity itself, which previous research has pointed to as a conceptual distinction that students, even university students who have taken 3 semesters of calculus, often need to revisit (Rasmussen, 2001).

The teacher's proactive role in advancing this idea involved not only continuing to act in accordance with certain social norms but involved what we see as a slide between being noninterventionist and assuming greater responsibility. Initially, as students offered their ideas and explanations for why the rate of change in should be 2 or  $2t$ , the teacher was fairly noninterventionist. He did not evaluate or otherwise try to direct students' thinking. However, as students explained their thinking, the teacher focused discussion on the distinction between rate of change in a quan-

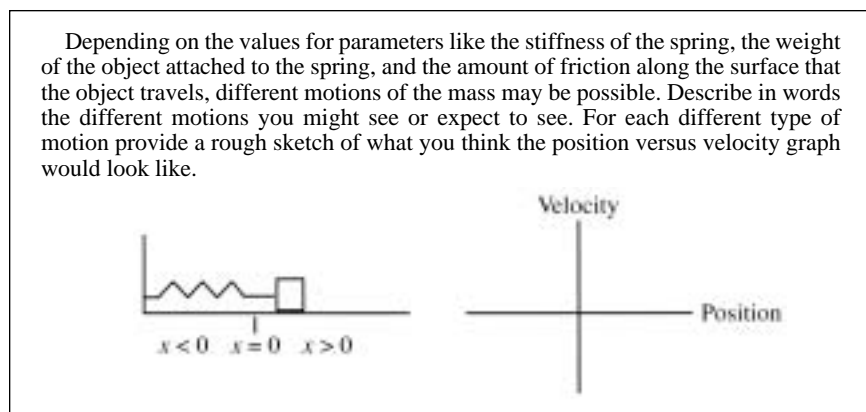
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<sup>7</sup> We do note, however, that Heath mentions a slope field, even though a slope field for this situation was never obtained. Indeed, students could not have obtained a slope field because they have yet to create the rate of change equation. As we argued in the transformational record section, slope fields initially emerged as a record of student reasoning and subsequently shifted to function as a means for reasoning about different mathematical ideas. Heath's use of a slope field in this instance is additional evidence that slope fields functioned for reasoning about new mathematical ideas.

tity and the quantity itself. Recall at one point the teacher said, “And, you told me that after  $t$  minutes there are  $2t$  pounds of salt flowing into the tank. Is that the rate of change?” We characterize his proactive efforts in pointing out this conceptual distinction as a slide toward assuming greater responsibility for articulating important mathematical ideas.

### *Example from Classroom B*

This final episode illustrates the proactive role of the teacher in strategically orchestrating a classroom discussion about a graph that she introduced after students had the opportunity to share with the whole class their position versus velocity graphs. This episode focuses on a task designed to introduce students to graphs in the phase plane. Prior to this activity, students engaged in a lesson intended to help them visualize solutions to systems of two differential equations in three dimensions and two-dimensional projections of these three-dimensional images. Consistent with the teacher’s typical pattern of asking students to generate and discuss mathematical ideas in small groups before leading a whole-class discussion, the teacher invited students to discuss and generate graphs for the scenario described in Figure 7.



*Figure 7. A spring-mass scenario.*

Students worked on this problem for some time in their small groups, and then the teacher invited Wes to share his group’s ideas with the rest of the class. Wes began by drawing an arc, as indicated in Figure 8, on the board. He explained his graph as follows:

**Wes:** OK. This is assuming no friction, gravity, loss of energy, anything. The system is constant and basically the velocity will be greatest when the spring crosses through its midpoint in its oscillation. In reality, it would do this [pointing to

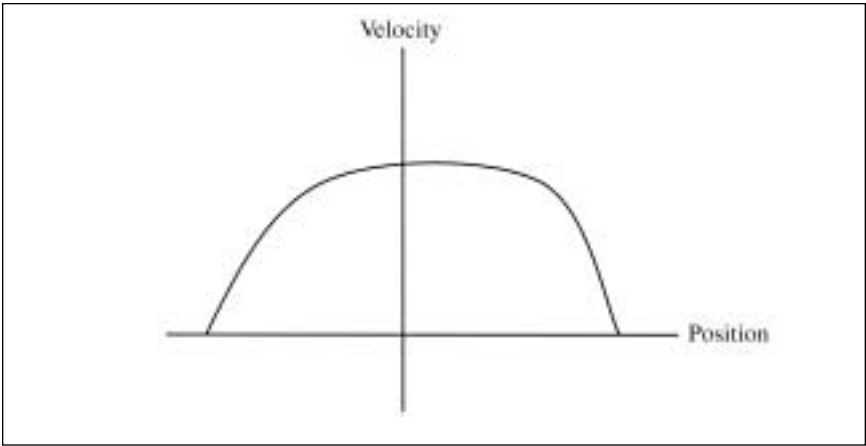


Figure 8. Wes’s sketch of a velocity versus position graph for a system with no friction.

the arc] once and then it would go down [draws smaller arcs, as in Figure 9]. It would get smaller and smaller due to friction, gravity, and all the other stuff. Any questions?

A number of students questioned why Wes’s graph did not include negative velocity, that is, why his graph lay only in the upper half plane. A discussion about whether or not Wes’s graph adequately displayed the spring-mass system’s negative velocity ensued. Jay volunteered his ideas at the board by drawing a circle in a clockwise direction, as shown in Figure 10. Jay reasoned within the context of

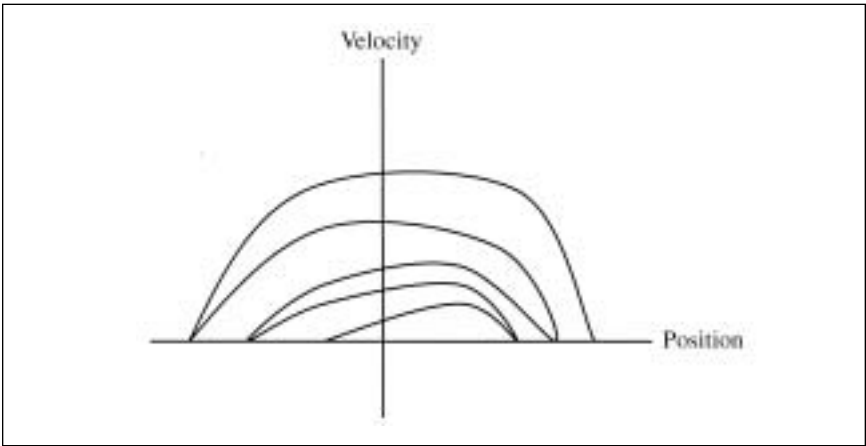


Figure 9. Wes’s sketch of a velocity versus position graph for a system with friction.



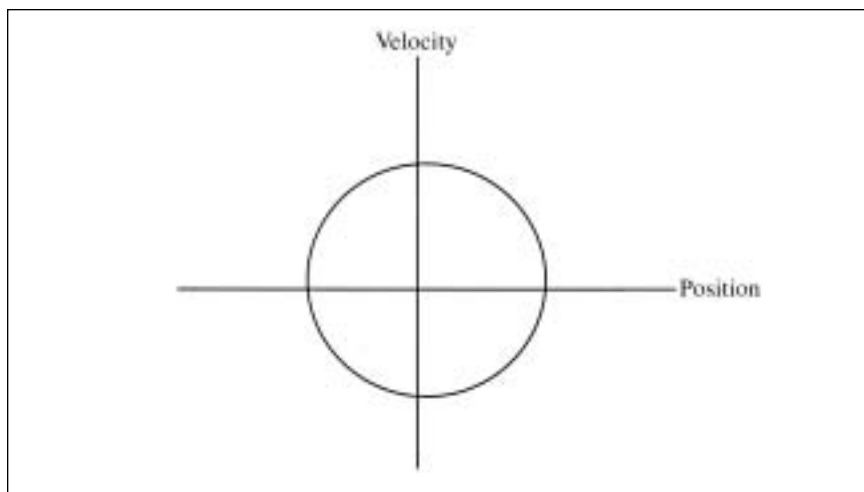


Figure 10. Jay's position versus velocity graph for a system with no friction.

the spring-mass system to convince Wes and others that Wes's graph adequately represented *speed* versus position, not velocity versus position.

During the discussion, Jay emphasized that his own graph did not include friction, unlike Wes's graph of smaller and smaller arcs. The class came to a consensus that Jay's graph more appropriately represented velocity versus position and that Wes's graph more appropriately represented speed versus position. At this point, the teacher addressed the class and posed the following question: "If we did introduce friction and we wanted to construct a velocity versus position graph, what would it look like?"

This question is significant for two reasons. First, the teacher assumed greater responsibility for the direction of the discussion, thus indicating a shift along the noninterventionist-total responsibility continuum. Second, the question built upon students' ideas and moved the mathematical agenda forward. Wes included friction in one of his graphs, and although Jay did not, he attended to it in his discussion. Thus, friction was an important idea in students' reasoning and gave the teacher an opportunity to make explicit connections between mathematics and physics.

In response to the teacher's question, Ryan sketched an inward-moving spiral, as shown in Figure 11, and reasoned as follows:

**Ryan:** As friction acts on it, you're going to get less velocity over time because friction's reducing your velocity so you're going to get less displacement over time. . . . It's going to lose a certain amount of velocity and a certain amount of displacement over time.

Ryan, like other students who explained their way of thinking before him, reasoned within the context of the spring-mass system and drew on his understanding of physics to present his argument. As evidence that the teacher's actions were produc-

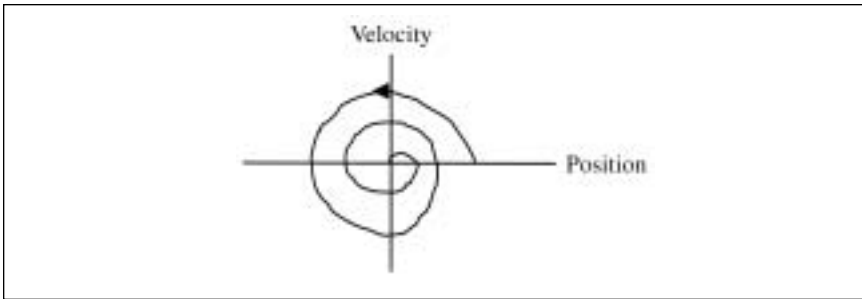


Figure 11. Ryan's graph of the spring-mass system with friction.

tive, we note that Hank followed Ryan's comments by adding, "If you took the absolute value of [Ryan's graph], you'd have exactly Wes's graph."

At this point the teacher could have solicited more graphs from the class or moved on to the next activity in the lesson. However, she posed the following question to the class:

*Teacher:* Let me ask you all a question. Can we have a graph like this? [She sketches the graph shown in Figure 12.] Is there a situation with the spring-mass system that would match that graph? It's a little bit different from the one that Ryan put up.

We point out that the teacher again assumed more responsibility for the direction of the class discussion and the mathematical content when she invited the class to consider the graph in Figure 12. The teacher's actions here are significant because she did not have to wait for students to offer diverse ideas; rather, she was able to draw upon her content expertise to further question students' thinking. This is consistent with the notion that teachers' mathematics expertise is an essential aspect of making pedagogical decisions (Ball 1991; Fennema & Franke, 1992).

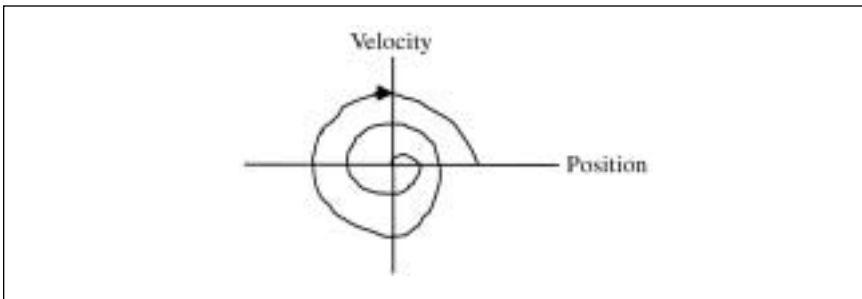


Figure 12. The teacher's outward spiraling graph.

A number of students offered explanations to justify the outward moving spiral graph. Among them, Kevin stated, “Add energy to the system,” and Josh offered, “If you were moving it [the mass and spring] on your own.” Thus, students offered justifications in terms of their understanding of what needed to be added to the physical system in order to produce an outward spiraling graph.

This example illustrated the proactive role of the teacher in strategically orchestrating classroom discussions around appropriate graphical representations of motion. We consider this an example of a generative alternative because the teacher-created graph contributed to the ongoing constitution of social norms pertaining to explanation and justification, and it furthered students’ emerging mathematical and physical reasoning about graphs of motion. We point out that in this example the ongoing constitution of the social norm that students provide explanations contributed to moving the mathematical agenda forward because as students provided explanations for the teacher’s outward spiraling graph, their reasoning about the physical system and its limitations became explicit. In providing these explanations, students broadened their reasoning about graphical representations of motion. This reasoning was used by the teacher in subsequent investigations of the spring-mass system, namely, leading to the development of straight line solutions. Finally, we point to the proactive role of the teacher in this example in terms of introducing a graph for class discussion, rather than waiting for a student to offer such an idea and her proactive role in continuing to act in accordance with the social norms regarding explanation and justification that she wished to foster.

## CONCLUSION

Part of the intent of the instructional design theory of RME is to inform the creation of sequences of tasks in which students develop important mathematical ideas and methods through solving a series of connected and challenging problems. This intention is captured in Freudenthal’s (1991) adage that, first and foremost, mathematics is a human activity. The heuristics of emergent models and guided reinvention offer the instructional designer a way to think about the nature of students’ mathematical activity and how a well-connected instructional sequence can be created to support students’ increasingly sophisticated ways of reasoning. Freudenthal’s adage has mostly been used to refer to how students should experience mathematics learning. We think that this adage applies equally well to the role of teachers. An important part of mathematics teaching is responding to student activity, listening to student activity, notating student activity, learning from student activity, and so on. In this sense, mathematics teaching is a human activity *about* human (i.e., student) activity. Framed in this way, the two PCTs we developed in this article function as resources for a teacher to think about his or her own teaching activity in relation to the students’ mathematical activity.

The RME heuristic of emergent models speaks to the long-term nature of a hypothetical learning trajectory (Gravemeijer, 1999). In comparison, the notion of transformational record falls within the realm of short-term or day-to-day teaching

practice. The examples we tendered illustrate how a transformational record can support a teacher's proactive role in furthering students' mathematical reasoning in ways that are increasingly compatible with the reasoning and symbolizing of the broader mathematical community. In the example from Classroom A, the teacher led a whole-class discussion about slope before he recorded the students' thinking. The teacher's decision to lead a whole-class discussion about slope enabled his vector notation to emerge as a record of students' collective thinking. In contrast, the teacher in Classroom B provided notation that fit with an individual student's thinking. The teacher's decision to inscribe one student's ideas early in the class discussion served the purpose of directing students' attention to the rate of change of solution graphs as well as of foreshadowing the mathematical idea of the phase line. The difference in when these teachers decided to notate student thinking underscores the fact that there is more than one way to instantiate the PCT of transformational record. PCTs, like any good tool, ought to be adaptable to local circumstances.

The PCT of generative alternative offers a teacher a way to think about how he or she can slide between a noninterventionist form of practice and a form of practice that assumes total responsibility for the mathematical content. In the example from Classroom A, the teacher began to assume more responsibility for the mathematical content and the direction of the whole-class discussion when he recognized that many students were not attending to an important conceptual distinction between rate of change in a quantity and the quantity itself. The teacher directed the classroom discussion to this issue and solicited students' elaboration of this difference. In the example from Classroom B, the teacher assumed more responsibility for facilitating whole-class discussion regarding her initiated alternative that functioned generatively to further students' reasoning about the graphical representation of motion. Thus, in both examples, the teacher maneuvered along the noninterventionist-total responsibility continuum.

The generative alternative construct, which is the teaching analogue to RME instructional design heuristic of guided reinvention, serves the dual function of furthering students' mathematical reasoning and contributing to the ongoing constitution of the norms of explaining and justifying one's thinking, listening to and attempting to make sense of other's thinking, and responding to challenges and questions. In each of our examples, the generative alternatives offered an occasion for students to provide explanations and justification for why they favored one option over another. In addition, both the mathematical agenda and students' mathematical reasoning were furthered through student reflection on their own thinking and the explanations of others.

An analysis of the classroom data points to the fact that generative alternatives can arise in two different ways. One way is that generative alternatives may be introduced to the class by the teacher with the intention that students will make explicit their reasoning about a concept or issue, as in Classroom B. A second way that a generative alternative may be introduced to the class is by a student's idea or an example that the teacher deliberately pursues. This was the case in the example from

Classroom A. In the case of Classroom A, the teacher intentionally initiated a discussion about the alternative in order to solicit justifications from the students. As with transformational records, there is more than one way for teachers to use the PCT of generative alternatives.

As evidenced in these four examples, discourse was a prominent feature of both classrooms. We argue that the teachers' use of transformational records and generative alternatives contributed to the development of a mathematical discourse community and thus created the "conditions for the possibility of learning" (Cobb, Boufi, McClain, & Whitenack, 1997, p. 264). For example, recall that a significant feature of generative alternative is that it functions as a teacher-driven mechanism to support students' evolving mathematical reasoning. The examples in both classrooms show the teacher providing instances (either by capitalizing on students' contributions or a deliberate introduction of an idea) for engaging students in argumentation. These assumptions about argumentation are consistent with Sfard's (2002) position that discourse is not a mathematical learning aid but rather constitutes learning (Lampert & Cobb, 2003; Sfard, 2002).

The notion of PCT can be viewed within the broader notion of cultural tool. Vygotsky (1978) introduced the notion of "cultural tool" as a means to mediate thought. Typically, cultural tools are verbal and conform to social conventions (Wertsch, 1991) but more recently have been extended to include such cultural artifacts as computer technology (Crawford, 1996). Individuals use cultural tools both to internalize others' ideas and to communicate one's own ideas. The development of concepts or ideas is thus inseparable from the cultural tools used to mediate thought. Vygotsky's notion of cultural tool is compatible with our notion of PCT in that students' ideas (which are typically communicated verbally) become functional in the classroom as the teacher supports the interactive development of meaning through the proactive use of various artifacts, such as graphical inscriptions and symbolic expressions. Vygotsky (1978) developed the notion of cultural tool within a broader sociocultural context, whereas we developed the construct of PCT in the microcosm of the classroom, a much more specific context. Both the cultural tool and the PCT are used to mediate social transactions, the former within society at large, and the latter within the classroom.

The PCTs of transformational record and generative alternative both move the mathematical agenda forward but in different ways. Generative alternatives promote discursive activity in which mathematical ideas are debated and claims are justified. Transformational records, on the other hand, complement discursive activity with symbolizing. In addition, both forms of PCTs contribute to the constitution of particular social norms regarding explanation and justification. Although we did not do so in this article, a complementary line of inquiry would be to examine the ways in which these PCTs function in the initiation and ongoing constitution of sociomathematical norms. For example, what role do these PCTs play in the initiation and constitution of what counts as an acceptable mathematical explanation? Are there other types of PCTs that help promote particular sociomathematical norms?

The explicit theoretical and practical aim of this article is to explicitly connect the instructional design theory of RME to teaching practices. We therefore make no claim to have exhausted the types of PCTs available to teachers. Certainly there are other types of PCTs that need to be explicated. For example, Burtch (2004) examined how an undergraduate teacher used thought experiments as a tool to support conjecturing as a normative activity and to move forward specific mathematical ideas. Thought experiments might well be another type of PCT.

As another example of a potential PCT, Staples (2004) studied a ninth-grade pre-algebra teacher, Ms. Nelson, who was particularly skillful in fostering collaborative inquiry in which her students learned mathematics with understanding. Staples referred to one of Ms. Nelson's productive teaching tools as "pressure points." Pressure points are verbal statements that build on students' current mathematical reasoning by pushing them to think in new ways. For example, when teaching the concept of function, Ms. Nelson pressed students to verbally explain in their own words a process for finding missing values in a table of values and then pressed them to verbalize a single relationship between the input and output values. Subsequent pressure points ultimately led students to write rules in terms of variables  $x$  and  $y$ . We view the notion of pressure points as a type of PCT because it blends pedagogical and mathematical expertise in a way that connects to students' thinking and moves forward the mathematical agenda.

We developed the PCTs of transformational record and generative alternative in the context of differential equations as a means to complement the RME instructional design heuristics of emergent models and guided reinvention. These two PCTs offer a vision of how teaching might proceed in a manner that reflects the underlying intention of RME. A two-part open question is the extent to which these two PCTs are valuable for (1) teachers who are using RME-inspired instructional materials in other content areas and (2) teachers who are not using RME-inspired instructional materials but who are developing student-centered instructional approaches.

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